Ising on the Cake

1 Abstract

The Ising model is one of physics's simplest yet most successful models. It consists of a d-dimensional lattice of binary spins (up/down), and a Hamiltonian. Probabilistic methods, such as Monte Carlo, are commonly used to come to numerical solutions.

Your task will be to study the thermodynamic behavior of Ising models of large systems using numerical methods. Solutions will demonstrate an understanding of the physics of Ising models, implementation of a numerical Ising model, and application of HPC skills (parallelism, numerical analysis, visualization, etc).

2 Background

The Ising model is most clearly understood in terms of ferromagnetic systems (though its application is much more abstract). Ferromagnetic materials can be thought of as consisting of small magnetized domains, each pointing in a particular direction. If the grains are randomly directed, there will be no noticeable large-scale magnetism (they all cancel). If the grains are in line, the material appears magnetic.

The Ising model restricts this understanding somewhat. The grains, or spins, are placed on a d dimensional lattice. Each spin in constrained to be in one of two states: spin up or spin down.

$$\sigma[\vec{x}] \in \{-1, 1\} \tag{1}$$

Hamiltonians tend to be quite simple, the nearest-neighbor Hamiltonian being the best example.

$$\mathcal{H} = \left(\sum_i \sum_j -J\sigma_i\sigma_j f(i,j) \right) + \left(\sum_i -h\sigma_i \right) \tag{2}$$

Here, J is the interaction strength, h is the strength of the external field, and $f(i,j)$ picks out 'nearest neighbors':

$$f(i,j) = \begin{cases} 1 & \text{if } \exists \delta \in S : i = j + \delta \\ 0 & \text{otherwise} \end{cases}$$

For instance, in two-dimensions, the lattice point $(1,1)$ would be nearest neighbors with $(0,1), (1,0), (1,2), (2,1)$

The first term (double sum) is an interaction energy: the neighboring spins want to line up with each-other. The second term (single sum) is the effect of the external field H: the spins want to line up with the field. The generalization to other dimensions should be clear. Hamiltonians can be made more complex by addition longer range or multi-spin terms.

Ising models are used to study a variety of phenomena. Examples include:

- Critical Temperature
- Correlation length
- Magnetization
- Energy
- Magnetic Suseptability
- Specific Heat
- Critical exponents $(\alpha, \beta, \gamma, \delta, \nu, \eta)$

3 Problem

The Ising model is well-studied, so finding a novel problem is difficult. Don't hesitate to look at prior work on the topic, but be sure to cite your sources in your solutions and demonstrate a personal understanding of the physics at hand.

Here are your objectives. You may assume nearest-neighbor Hamiltonians with arbitrary non-negative values of J and h (eq. 2).

1. Derive the mean-field approximation (MFA) solution in 1-dimension.
2. Implement the 2 and 3 dimensional Ising model
3. Run your implementation on an assortment of lattice sizes and comment on the performance of the code.
4. Find the critical temperature in 2 and 3 dimensions for the cases $(J = 1, h = 0), (J = 1, h = 1), (J = 1, h = 2), (J = a, h = b : a, b \geq 0)$. Compare your results to published values when possible.
5. Find the magnetization, energy, magnetic susceptibility, and specific heat in the neighborhood of the critical temperature for $(J = 1, h = 0)$.
6. Compare the 2 and 3 dimensional Ising models with respect to your findings.
7. Compare your results for an appropriate assortment of lattice sizes.
8. Propose and implement an alternative Hamiltonian. Discuss its implications.

4 Submission/Grading

Your submission should include:

- Report of your findings with citations where appropriate
- Source code
- Compiled binary
- Usage manual
- Any additional materials

5 Recommended Readings

- `http://en.wikipedia.org/wiki/Ising_model`
- `http://www.pha.jhu.edu/~javalab/ising/ising.html`
- `http://farside.ph.utexas.edu/teaching/329/lectures/node110.html`